

The Moore-Penrose inverse of a companion matrix*

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Abstract

Necessary and sufficient conditions are given for the Moore-Penrose inverse of a companion matrix over an arbitrary ring to exist.

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1 Introduction

Let R be a ring with 1 and involution $(\bar{\cdot})$. That is, for all $a, b \in R$, the equalities $\bar{\bar{1}} = 1$, $\overline{(a+b)} = \bar{a} + \bar{b}$ and $\overline{(ab)} = \bar{b}\bar{a}$ hold. The involution $(\bar{\cdot})$ in R endows an involution $*$ in the set $\mathcal{M}(R)$ of (finite) matrices over R , defined as $[a_{ij}]^* = [\bar{a}_{ji}]$.

A matrix A is said to be Moore-Penrose invertible with respect to $*$ provided there is A^\dagger such that

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, (A^\dagger A)^* = A^\dagger A.$$

If such a matrix A^\dagger exists, then it is well known it is unique (see [1]).

We say $a \in R$ is regular if $a \in aRa$, or equivalently $axa = a$ is a ring consistent equation. A particular solution is denoted by a^- and called a von Neumann inverse of a . A regular ring is a ring whose elements are regular. It is a standard fact that if R is a regular ring then the ring of $m \times m$ matrices over R is again regular (see, for instance, [2]).

We will use the following known fact:

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Lemma 1.1. *Given $x, y \in R$, then $1 + xy$ is a unit if and only if $1 + yx$ is a unit, and in this case*

$$(1 + xy)^{-1} = 1 - x(1 + yx)^{-1}y.$$

Lemma 1.1 has a useful extension for rectangular matrices which we will need later on. Given $n \times k$ matrices B and C , then

$$I_n + BC^T \text{ is invertible if and only if } I_k + C^T B \text{ is invertible.} \quad (1)$$

Versions of this relation for generalized inverses can be found in [3] and [4].

Using von Neumann inverses, it was shown in [6], [8], [9] how to characterize the existence of a Moore-Penrose inverse by means of units. The equivalence between the existence of M^\dagger , the invertibility of $U = MM^* + I - MM^-$, and the invertibility of $V = M^*M + I - M^-M$ will play an important role throughout this paper.

Theorem 1.1. *Let $a \in R$ be a regular element, and a^- a von Neumann inverse of a . The following conditions are equivalent:*

- (a) a^\dagger exists;
- (b) $s = a\bar{a}aa^- + 1 - aa^-$ is a unit;
- (c) $h = a^-a\bar{a}a + 1 - a^-a$ is a unit;
- (d) $v = \bar{a}a + 1 - a^-a$ is a unit;
- (e) $u = a\bar{a} + 1 - aa^-$ is a unit.

In this case,

$$a^\dagger = \overline{(s^{-1}a)} = \overline{(ah^{-1})} = \overline{(u^{-1}a)} = \overline{(av^{-1})}.$$

Proof. The equivalences (a) \Leftrightarrow (b) \Leftrightarrow (c) follow from [8, Theorem 2], as well as the first two expressions for a^\dagger .

(b) \Leftrightarrow (d). Write $s = a\bar{a}aa^- + 1 - aa^- = 1 - a(-\bar{a}aa^- + a^-) = 1 - yx$ with $y = a$ and $x = -\bar{a}aa^- + a^-$. Then $v = 1 - xy = \bar{a}a + 1 - a^-a$ and the equivalence follows using Lemma 1.1, with $v^{-1} = 1 + xs^{-1}y = 1 + (-\bar{a}aa^- + a^-)s^{-1}a = a^-s^{-1}a + 1 - \bar{a}s^{-1}a$ and $s^{-1} = (1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x = 1 + yv^{-1}x = 1 + av^{-1}(-\bar{a}aa^- + a^-) = av^{-1}a^- + 1 - av^{-1}\bar{a}aa^-$.

(c) \Leftrightarrow (e). Now, write $u = a\bar{a} + 1 - aa^- = 1 - a(-a^-a\bar{a} + a^-) = 1 - xy$ with $x = a$ and $y = -a^-a\bar{a} + a^-$. Then $h = 1 - yx = 1 - (-a^-a\bar{a} + a^-)a = a^-a\bar{a}a + 1 - a^-a$ and the equivalence follows using Lemma 1.1, with $u^{-1} = 1 + xh^{-1}y = 1 + ah^{-1}(-a^-a\bar{a} + a^-) = ah^{-1}a^- + 1 - ah^{-1}a^-a\bar{a} = ah^{-1}a^- + 1 - ah^{-1}\bar{a}$ and $h^{-1} = (1 - yx)^{-1} = 1 + y(1 - xy)^{-1}x = 1 + (-a^-a\bar{a} + a^-)u^{-1}a = a^-u^{-1}a + 1 - a^-a\bar{a}u^{-1}a$.

We now derive the expressions for the a^\dagger . From [8, Theorem 2], $a^\dagger = \overline{(ah^{-1})}$. Since

$$\begin{aligned} ah^{-1} &= aa^{-}u^{-1}a + a - a\bar{a}u^{-1}a \\ &= aa^{-}u^{-1}a + uu^{-1}a - a\bar{a}u^{-1}a \\ &= (aa^{-} - a\bar{a} + u)u^{-1}a \\ &= u^{-1}a \end{aligned}$$

then $a^\dagger = \overline{(u^{-1}a)}$.

Finally, since $ua = a\bar{a}a = av$ then $u^{-1}a = av^{-1}$ and $a^\dagger = \overline{(av^{-1})}$. \square

Consider the $(n+1) \times (n+1)$ companion matrix

$$M = \begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix},$$

with $a \in R$ and $\mathbf{b} \in R^n$. In this paper, we are interested on characterizing the existence of M^\dagger by means of units in R . For the group inverse of M the reader is referred to [5] and [10].

We will reduce the Moore-Penrose inverse of the companion matrix M to the lower triangular case, by using the factorization $M = AP$ where

$$A = \begin{bmatrix} a & 0 \\ \mathbf{b} & I_n \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & 1 \\ I_n & 0 \end{bmatrix}.$$

Since M is unitarily equivalent to A , then M has a Moore-Penrose inverse exactly when A is Moore-Penrose invertible. Furthermore,

$$M^\dagger = P^* A^\dagger.$$

In this paper, we will assume a to be regular in R , that is, there exists $a^- \in R$ for which $aa^-a = a$. Given solutions (possibly distinct) $a^-, a^=$ to $axa = a$ in R , then one can construct a reflexive inverse of a , that is, a common solution to $axa = a$ and $xax = x$, by taking $a^+ = a^-aa^-$.

Note that the Moore-Penrose invertibility of A does not imply a is Moore-Penrose invertible. Indeed, consider R the ring of 2×2 complex matrices with transposition as the involution, and set

$$a = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & 0 \\ \mathbf{b} & I_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right].$$

Using [11], A^\dagger exists since $rk(A) = rk(AA^T) = rk(A^T A) = 3$, but a^\dagger does not since $aa^T = 0$. Consequently, the Moore-Penrose invertibility of the companion matrix M does not imply the Moore-Penrose invertibility of a .

On the other hand, a may be Moore-Penrose invertible and M^\dagger may not exist. As an example, consider R the ring of 2×2 complex matrices with transposition as the involution,

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a^\dagger, \mathbf{b} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, M = \begin{bmatrix} 0 & a \\ I_2 & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & i \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then $rk(M) = 3 \neq 2 = rk(M^T M)$ for M as a 4×4 complex matrix, and M^\dagger does not exist.

We will divide this paper in two parts. In the first, we will assume a^\dagger exists, and in the second we just assume regularity of a .

2 The case a^\dagger exists

Suppose a^\dagger exists and consider the unit

$$u = a\bar{a} + 1 - aa^\dagger, \text{ with } u^{-1} = \bar{a}^\dagger a^\dagger + 1 - aa^\dagger.$$

Note that $u^{-1}a = \bar{a}^\dagger$ and $\bar{a}u^{-1} = a^\dagger$.

The matrix

$$A = \begin{bmatrix} a & 0 \\ \mathbf{b} & I_n \end{bmatrix}$$

is Moore-Penrose invertible if and only if $U = AA^* + I_{n+1} - AA^-$ is invertible for one, and hence, all choices of von Neumann inverses A^- of A , by Theorem 1.1. Applying [7, Theorem 1], we may take

$$A^- = \begin{bmatrix} a^\dagger & 0 \\ -\mathbf{b}a^\dagger & I_n \end{bmatrix},$$

for which choice we obtain

$$AA^- = \begin{bmatrix} aa^\dagger & 0 \\ 0 & I_n \end{bmatrix}$$

and

$$U = \begin{bmatrix} a\bar{a} + 1 - aa^\dagger & a\mathbf{b}^* \\ \mathbf{b}\bar{a} & \mathbf{b}\mathbf{b}^* + I_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{b}\bar{a}u^{-1} & I_n \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} 1 & u^{-1}a\mathbf{b}^* \\ 0 & I_n \end{bmatrix},$$

where

$$\begin{aligned} Z &= \mathbf{b}\mathbf{b}^* + I_n - \mathbf{b}\bar{a}u^{-1}a\mathbf{b}^* \\ &= I_n + \mathbf{b}(1 - \bar{a}u^{-1}a)\mathbf{b}^* \\ &= I_n + \mathbf{b}(1 - a^\dagger a)\mathbf{b}^* \end{aligned}$$

Now, the invertibility of Z is equivalent to $z = 1 + \mathbf{b}^* \mathbf{b} (1 - a^\dagger a)$ being a unit of R , by the equivalence (1). Writing $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, this is the same as

$$z = 1 + \sum_{i=1}^n \bar{b}_i b_i (1 - a^\dagger a) \quad (2)$$

being a unit of R .

Theorem 2.1. *Given $a \in R$ such that a^\dagger exists and $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, then the following are equivalent:*

- (a) *The companion matrix $M = \begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}$ is Moore-Penrose invertible.*
- (b) *$1 + (1 - a^\dagger a) \mathbf{b}^* \mathbf{b} (1 - a^\dagger a)$ is a unit of R .*
- (c) *$1 + \mathbf{b}^* \mathbf{b} (1 - a^\dagger a)$ is a unit of R .*
- (d) *$1 + (1 - a^\dagger a) \mathbf{b}^* \mathbf{b}$ is a unit of R .*

We now carry out the construction of the Moore-Penrose inverse of the companion matrix, in the case a^\dagger exists.

Using Theorem 1.1,

$$A^\dagger = (U^{-1}A)^*$$

which leads to

$$\begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}^\dagger = \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} A^\dagger = \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} (U^{-1}A)^*.$$

Note that u , U and Z are symmetric, and hence also are their inverses. Therefore,

$$\begin{aligned} U^{-1} &= \begin{bmatrix} 1 & -\bar{a}^\dagger \mathbf{b}^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{b} a^\dagger & I_n \end{bmatrix} \\ &= \begin{bmatrix} u^{-1} + \bar{a}^\dagger \mathbf{b}^* Z^{-1} \mathbf{b} a^\dagger & -\bar{a}^\dagger \mathbf{b}^* Z^{-1} \\ -Z^{-1} \mathbf{b} a^\dagger & Z^{-1} \end{bmatrix}, \end{aligned}$$

with $Z^{-1} = (I_n + \mathbf{b} (1 - a^\dagger a) \mathbf{b}^*)^{-1} = I_n - \mathbf{b} (1 - a^\dagger a) z^{-1} (1 - a^\dagger a) \mathbf{b}^*$ and $z = 1 + (1 - a^\dagger a) \mathbf{b}^* \mathbf{b} (1 - a^\dagger a)$. Then

$$\begin{aligned} A^\dagger &= A^* (U^*)^{-1} \\ &= \begin{bmatrix} \bar{a} u^{-1} + a^\dagger a \mathbf{b}^* Z^{-1} \mathbf{b} a^\dagger - \mathbf{b}^* Z^{-1} \mathbf{b} a^\dagger & -a^\dagger a \mathbf{b}^* Z^{-1} + \mathbf{b}^* Z^{-1} \\ -Z^{-1} \mathbf{b} a^\dagger & Z^{-1} \end{bmatrix} \\ &= \begin{bmatrix} a^\dagger - (1 - a^\dagger a) \mathbf{b}^* Z^{-1} \mathbf{b} a^\dagger & (1 - a^\dagger a) \mathbf{b}^* Z^{-1} \\ -Z^{-1} \mathbf{b} a^\dagger & Z^{-1} \end{bmatrix}. \end{aligned}$$

Finally,

$$\begin{aligned} \begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}^\dagger &= \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} A^\dagger \\ &= \begin{bmatrix} -Z^{-1}\mathbf{b}a^\dagger & Z^{-1} \\ a^\dagger - (1 - a^\dagger a)\mathbf{b}^*Z^{-1}\mathbf{b}a^\dagger & (1 - a^\dagger a)\mathbf{b}^*Z^{-1} \end{bmatrix}. \end{aligned}$$

3 The case a is regular

We note that the companion matrix

$$\begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}$$

is regular if and only if a is regular. This follows from the factorization

$$\begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} I_n & \mathbf{b} \\ 0 & 1 \end{bmatrix}.$$

Suppose a is regular and let a^+ be any reflexive inverse of a .

The matrix

$$A = \begin{bmatrix} a & 0 \\ \mathbf{b} & I_n \end{bmatrix}$$

is Moore-Penrose invertible if and only if $V = A^*A + I_{n+1} - A^-A$ is invertible for one, and hence, all choices of von Neumann inverses A^- of A , by Theorem 1.1. Applying [7, Theorem 1], we may take

$$A^- = \begin{bmatrix} a^+ & 0 \\ -\mathbf{b}a^+ & I_n \end{bmatrix},$$

for which choice we obtain

$$A^-A = \begin{bmatrix} a^+a & 0 \\ -\mathbf{b}a^+a + \mathbf{b} & I_n \end{bmatrix}$$

and

$$V = \begin{bmatrix} \bar{a}a + 1 - a^+a + \mathbf{b}^*\mathbf{b} & \mathbf{b}^* \\ \mathbf{b}a^+a & I_n \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{b}^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{b}a^+a & I_n \end{bmatrix},$$

where

$$\begin{aligned} \zeta &= \bar{a}a + 1 - a^+a + \mathbf{b}^*\mathbf{b}(1 - a^+a) \\ &= \bar{a}a + 1 - a^+a + \sum_{i=1}^n \bar{b}_i b_i (1 - a^+a), \end{aligned}$$

with $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.

Theorem 3.1. *Given $a \in R$ and $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, then the companion matrix $M = \begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}$ is Moore-Penrose invertible if and only if a is regular and, for some reflexive inverse a^+ of a , the element*

$$\zeta = \bar{a}a + 1 - a^+a + \sum_{i=1}^n \bar{b}_i b_i (1 - a^+a) \quad (3)$$

is a unit of R .

We now construct the Moore-Penrose inverse of the companion matrix, in the case a is regular.

Using Theorem 1.1, the Moore-Penrose inverse of A is given by

$$A^\dagger = (AV^{-1})^*$$

where

$$\begin{aligned} V^{-1} &= (A^*A + I_{n+1} - A^-A)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -\mathbf{b}a^+a & I_n \end{bmatrix} \begin{bmatrix} \zeta^{-1} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{b}^* \\ 0 & I_n \end{bmatrix} \end{aligned}$$

and $\zeta = \bar{a}a + 1 - a^+a + \mathbf{b}^*\mathbf{b}(1 - a^+a)$. Then

$$(V^{-1})^* = \begin{bmatrix} \bar{\zeta}^{-1} & -\bar{\zeta}^{-1}\overline{(a^+a)}\mathbf{b}^* \\ -\mathbf{b}\bar{\zeta}^{-1} & I_n + \mathbf{b}\bar{\zeta}^{-1}\overline{(a^+a)}\mathbf{b}^* \end{bmatrix}.$$

Substituting in the expression of A^\dagger ,

$$\begin{aligned} A^\dagger &= (V^{-1})^* A^* \\ &= \begin{bmatrix} \bar{\zeta}^{-1} & -\bar{\zeta}^{-1}\overline{(a^+a)}\mathbf{b}^* \\ -\mathbf{b}\bar{\zeta}^{-1} & I_n + \mathbf{b}\bar{\zeta}^{-1}\overline{(a^+a)}\mathbf{b}^* \end{bmatrix} \begin{bmatrix} \bar{a} & \mathbf{b}^* \\ 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} \bar{\zeta}^{-1}\bar{a} & \bar{\zeta}^{-1}(1 - \overline{(a^+a)})\mathbf{b}^* \\ -\mathbf{b}\bar{\zeta}^{-1}\bar{a} & I_n - \mathbf{b}\bar{\zeta}^{-1}(1 - \overline{(a^+a)})\mathbf{b}^* \end{bmatrix} \end{aligned}$$

from which we deduce

$$\begin{aligned} \begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}^\dagger &= \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} A^\dagger \\ &= \begin{bmatrix} -\mathbf{b}\bar{\zeta}^{-1}\bar{a} & I_n - \mathbf{b}\bar{\zeta}^{-1}(1 - \overline{(a^+a)})\mathbf{b}^* \\ \bar{\zeta}^{-1}\bar{a} & \bar{\zeta}^{-1}(1 - \overline{(a^+a)})\mathbf{b}^* \end{bmatrix} \end{aligned}$$

4 Questions and remarks

1. If a^\dagger exists and $\mathbf{b}^*\mathbf{b} \in a^\dagger\mathbf{b}^*\mathbf{b}R$ then $\begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}^\dagger$ exists. Indeed, if $\mathbf{b}^*\mathbf{b} = a^\dagger\mathbf{b}^*\mathbf{b}x$ for some x in R then $a^\dagger a\mathbf{b}^*\mathbf{b} = a^\dagger\mathbf{b}^*\mathbf{b}x = \mathbf{b}^*\mathbf{b}$.
2. If a is regular and $b_i \in Rb_i a$ then $\begin{bmatrix} 0 & a \\ I_n & \mathbf{b} \end{bmatrix}^\dagger$ exists, with $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, if and only if a^\dagger exists. Indeed, if $b_i = xb_i a$ then $b_i a^+ a = xb_i a = b_i$, from which the element ζ in equation (3) collapses to $\zeta = \bar{a}a + 1 - a^+a$, which is a unit exactly when a^\dagger exists.
3. How can the invertible elements defined in equations (2) and (3) be directly related, in the case a^\dagger exists?

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